Math 2050A Test 2, 11 Nov

Answer the following four questions

1. (a) State without proof the Bolzano-Weierstrass Theorem (5);
   (b) State the definition of Cauchy sequence (5);
   (c) Prove that a sequence $\{x_n\}_{n=1}^{\infty}$ of real number is convergent if and only if it is a Cauchy sequence. (10)

   Solution:

   (a) If a sequence $\{x_n\}$ is bounded, then there exists a subsequence of $\{x_n\}$ that is convergent.

   (b) A sequence is a Cauchy sequence if $\forall \epsilon > 0$, $\exists N$, $\forall m, n > N$, $|x_m - x_n| < \epsilon$.

   (c) Suppose that $\{x_n\}$ is convergent and denote $a = \lim x_n$. Then $\forall \epsilon > 0$, $\exists N$, $\forall n > N$, $|x_n - a| < \epsilon/2$. For $m, n > N$, $|x_m - x_n| \le |x_n - a| + |x_m - a| < \epsilon$. $\{x_n\}$ is Cauchy.

   Suppose that $\{x_n\}$ is Cauchy. Then $\exists N_1$ such that $\forall m, n > N_1$, $|x_m - x_n| < 1$. Take $m = N_1 + 1$. Then $\forall n > N_1$, $|x_n - x_{N_1+1}| < 1$, or equivalently, $x_{N_1+1} - 1 < x_n < x_{N_1+1} + 1$. Let $M_1 = \min\{x_1, x_2, ..., x_{N_1}, x_{N_1+1} + 1\}$, $M_2 = \max\{x_1, x_2, ..., x_{N_1}, x_{N_1+1} + 1\}$. Then $M_1 \le x_n \le M_2$, $\forall n$. We know that $\{x_n\}$ is bounded. By Bolzano-Weierstrass Theorem, there is a subsequence $\{x_{n_k}\}$ that is convergent. Denote $a = \lim x_{n_k}$.

   Now fix $\epsilon > 0$. Then $\exists N_1$, $\forall k > N_1$, $|x_{n_k} - a| < \epsilon/2$. Since $\{x_n\}$ is Cauchy, $\exists N_2$, $\forall k, n > N_2$, $|x_k - x_n| < \epsilon/2$. Without loss of generality, we may assume that $N_2 > N_1$. Since $n_k \ge k$, we also have $|x_{n_k} - x_n| < \epsilon/2$ and $|x_n - a| \le |x_{n_k} - a| + |x_n - x_{n_k} < \epsilon$. Hence $\{x_n\}$ is convergent.

2. Using ε-δ terminology or the sequential criterion to show that

   (a) $\displaystyle\lim_{x\to 2} \frac{x^2+2}{x^2-1} = 2$ (10);

   (b) $\displaystyle\lim_{x\to 1, x>1} \frac{x^2+2}{x^2-1} = +\infty$ (10);

   (c) $\displaystyle\lim_{x\to +\infty} \frac{x^2+2}{x^2-1} = 1$ (10).

   Solution:

   (a) $|\frac{x^2+2}{x^2-1} - 2| = |\frac{x+2}{x^2-1}||x-2|$. When $|x-2| < \frac{1}{2}$, $|\frac{x+2}{x^2-1}| < \frac{18}{5}$. For $\epsilon > 0$, let $\delta = \min\{\frac{1}{2}, \frac{5}{18}\epsilon\}$. Then $|x-2| < \delta$ implies that $|\frac{x^2+2}{x^2-1} - 2| < \epsilon$. Hence $\displaystyle\lim_{x\to 2} \frac{x^2+2}{x^2-1} = 2$.

(b) $\frac{x^2+2}{x^2-1} = \frac{x^2+2}{x+1}\frac{1}{x-1}$. When $1 < x < 2$, $\frac{x^2+2}{x+1} > 1$. For $E > 0$, let $\delta = \min\{1, \frac{1}{E}+1\}$. Then $0 < x-1 < \delta$ implies that $\frac{x^2+2}{x^2-1} > E$. Hence $\lim_{x\to 1, x>1} \frac{x^2+2}{x^2-1} = +\infty$.

(c) $|\frac{x^2+2}{x^2-1} - 1| = |\frac{3}{x+1}||\frac{1}{x-1}| < \frac{3}{2(x-1)}$ when $x > 1$. For $\epsilon > 0$, let $M = \max\{1, \frac{3}{2\epsilon}+1\}$. Then $x > M$ implies that $|\frac{x^2+2}{x^2-1} - 1| < \epsilon$. Hence $\lim_{x\to+\infty} \frac{x^2+2}{x^2-1} = 1$.

3. Suppose $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of real numbers.

(a) Suppose $0 \le |a_n| < b_n$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} b_n$ is convergent, show that $\sum_{n=1}^{\infty} a_n$ is convergent (10). (Remark: This is called the absolute convergence test.)

(b) Prove the Dirichlet test by establishing the followings:

(i) (10) Show by mathematical induction that for $n \ge 2$, we have

$$\sum_{k=1}^{n} a_k(b_{k+1} - b_k) = a_n b_{n+1} - a_1 b_1 - \sum_{k=2}^{n} b_k(a_k - a_{k-1}).$$

(ii) (10) Suppose a_n is monotonic non-increasing (i.e. $a_{n+1} \le a_n$ for all $n \in \mathbb{N}$), $\lim_{n\to+\infty} a_n = 0$ and there exists $M > 0$ so that $|\sum_{k=1}^{n} b_k| \le M$ for all $n \in \mathbb{N}$. By using (a) and (b-i), show that the series $\sum_{n=1}^{\infty} a_n b_n$ converges. (Hint: Write $\sum_{k=1}^{n} a_k b_k$ as $\sum_{k=1}^{n} a_k(B_k - B_{k-1})$ where $B_m = \sum_{k=1}^{m} b_k$ for $m \in \mathbb{N}$ and $B_0 = 0$.)

Solution:

(a) Let $A_n = \sum_{k=1}^{n} a_k$ and $B_n = \sum_{k=1}^{n} b_k$. Since B_n is convergent, it is Cauchy, and $\forall \epsilon > 0$, $\exists N$, $\forall m > n > N$, $|B_m - B_n| = b_{n+1} + \cdots + b_m < \epsilon$, and then $|A_m - A_n| = |a_{n+1} + \cdots + a_m| \le |a_{n+1}| + \cdots + |a_m| \le b_{n+1} + \cdots + b_m < \epsilon$. $\{A_n\}$ is Cauchy and thus converges.

(b) (i) For $n = 2$, $a_1(b_2 - b_1) + a_2(b_3 - b_2) = a_2 b_3 - a_1 b_1 - b_2(a_2 - a_1)$. Suppose that the equality holds for $n = m$, then for $n = m+1$,

$$\begin{aligned}
\sum_{k=1}^{m+1} a_k(b_{k+1} - b_k) &= \sum_{k=1}^{m} a_k(b_{k+1} - b_k) + a_{m+1}(b_{m+2} - b_{m+1}) \\
&= a_{m+1} b_{m+2} - a_1 b_1 - b_{m+1}(a_{m+1} - a_m) - \sum_{k=2}^{m} b_k(a_k - a_{k-1}) \\
&= a_{m+1} b_{m+2} - a_1 b_1 - \sum_{k=2}^{m+1} b_k(a_k - a_{k-1}).
\end{aligned}$$

(ii) We calculate that

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &= \sum_{k=1}^{n} a_k(B_k - B_{k-1}) \\ &= a_n B_n - a_1 B_0 - \sum_{k=2}^{n} B_{k-1}(a_k - a_{k-1}) \\ &= a_n B_n - \sum_{k=2}^{n} B_{k-1}(a_k - a_{k-1})\end{aligned}$$

since $B_0 = 0$. Now B_n is bounded and $\lim a_n = 0$, so $\lim a_n B_n = 0$ (why?) Note that $|B_{k-1}(a_k - a_{k-1})| < M(a_{k-1} - a_k)$ and

$$\sum_{k=2}^{\infty} M(a_{k-1} - a_k) = Ma_1 - \lim_{n\to\infty} Ma_n = Ma_1.$$

By (i) we know that $\sum_{k=2}^{\infty} B_{k-1}(a_k - a_{k-1})$ is convergent. Hence $\sum_{k=1}^{\infty} a_k b_k$ is convergent.

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f has a limit L as $x \to 0$. Show that $L = 0$ and f has a limit at every $c \in \mathbb{R}$.

   Solution:

   Repeatedly applying the condition $f(x+y) = f(x) + f(y)$ we obtain $f(1) = f(\frac{1}{n} + \cdots + \frac{1}{n}) = nf(\frac{1}{n})$, or $f(\frac{1}{n}) = \dfrac{f(1)}{n}$, $\forall n \in \mathbb{N}$. If f has a limit L as $x \to 0$, then $L = \lim_{n\to\infty} f(\frac{1}{n}) = \lim_{n\to\infty} \dfrac{f(1)}{n} = 0$ since $f(1)$ is a constant.

   From $f(x+y) = f(x) + f(y)$ we also have $f(x) - f(c) = f(x-c)$. Now for $\epsilon > 0$, $\exists \delta > 0$ such that $|x - c| < \delta$ implies that $|f(x-c) - 0| = |f(x-c)| < \epsilon$, or equivalently, $|f(x) - f(c)| < \epsilon$. That is, $\lim_{x\to c} f(x) = f(c)$.